

Diagonalization

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Review

- If $Av = \lambda v$ (v is a vector, λ is a scalar)
 - v is an eigenvector of A **excluding zero vector**
 - λ is an eigenvalue of A that corresponds to v

- Eigenvectors corresponding to λ are **nonzero** solution of $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$

Eigenvectors

corresponding to λ

$$= \underline{\text{Null}(A - \lambda I_n)} - \{\mathbf{0}\}$$

eigenspace

Eigenspace of λ :

Eigenvectors

corresponding to $\lambda + \{\mathbf{0}\}$

- A scalar t is an eigenvalue of A



$$\det(A - tI_n) = 0$$

Review

- Characteristic polynomial of A is

$$\det(A - tI_n) \quad \text{Factorization}$$

multiplicity

$$= (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \dots (t - \lambda_k)^{m_k} (\dots \dots)$$

Eigenvalue:

λ_1

λ_2

λ_k

Eigenspace:

d_1

d_2

d_k

(dimension)

$\leq m_1$

$\leq m_2$

$\leq m_k$

Outline

- An $n \times n$ matrix A is called **diagonalizable** if $A = PDP^{-1}$
 - D : $n \times n$ diagonal matrix
 - P : $n \times n$ invertible matrix
- Is a matrix A **diagonalizable**?
 - If yes, find D and P
- Reference: Textbook 5.3

Diagonalizable

- An $n \times n$ matrix A is called **diagonalizable** if $A = PDP^{-1}$

- D : $n \times n$ diagonal matrix
- P : $n \times n$ invertible matrix


$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- **Not all matrices are diagonalizable**

➔ $A^2 = 0$ (?)

If $A = PDP^{-1}$ for some invertible P and diagonal D

➔ $A^2 = PD^2P^{-1} = 0$ ➔ $D^2 = 0$ ➔ $D = 0$

➔ $A = 0$ 

D is diagonal

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

$$A = PDP^{-1} \quad \longrightarrow \quad AP = PD$$

$$\longrightarrow AP = [\underline{Ap_1} \quad \cdots \quad \underline{Ap_n}]$$

$$\longrightarrow PD = P[d_1e_1 \quad \cdots \quad d_n e_n]$$

$$= [Pd_1e_1 \quad \cdots \quad Pd_n e_n]$$

$$= [d_1Pe_1 \quad \cdots \quad d_nPe_n]$$

$$= [\underline{d_1p_1} \quad \cdots \quad \underline{d_n p_n}] \quad \longrightarrow \quad Ap_i = d_i p_i$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

Diagonalizable

- If A is diagonalizable

$$A = PDP^{-1}$$

||

There are n eigenvectors that form an invertible matrix

||

There are n independent eigenvectors

||

The eigenvectors of A can form a basis for \mathbb{R}^n .

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

p_i is an eigenvector of A
corresponding to eigenvalue d_i

Diagonalizable

- If A is diagonalizable

$$A = PDP^{-1}$$

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

p_i is an eigenvector of A
corresponding to eigenvalue d_i

How to diagonalize a matrix A ?

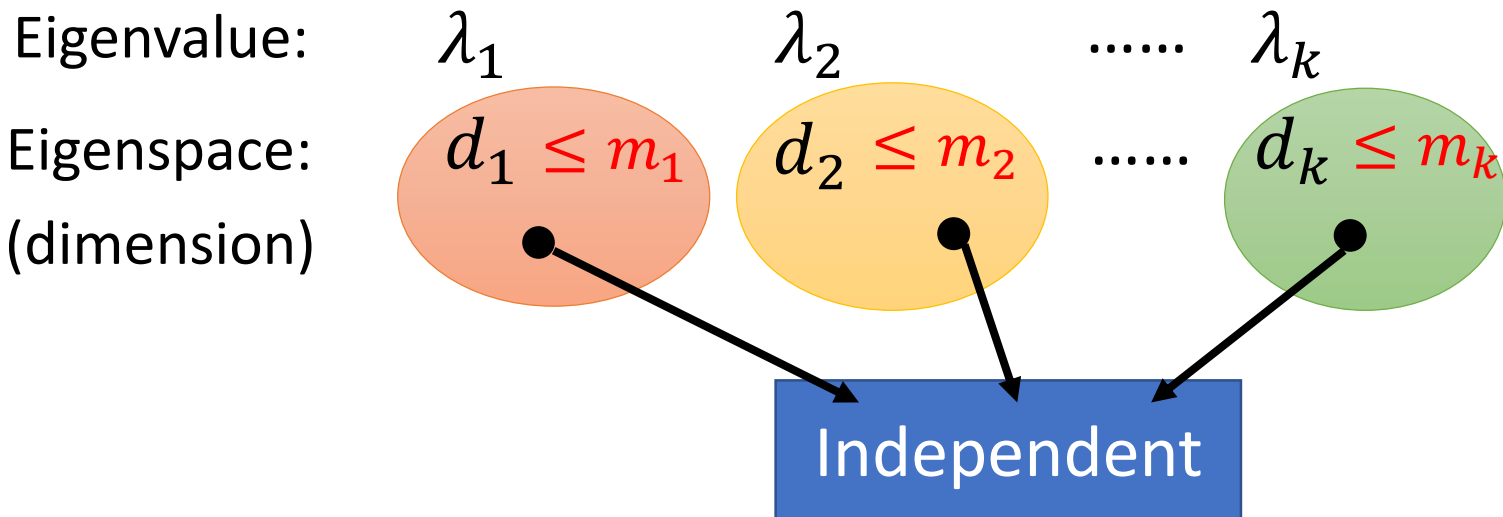
- Step 1: Find n L.I. eigenvectors corresponding if possible, and form an invertible P
- Step 2: The eigenvalues corresponding to the eigenvectors in P form the diagonal matrix D .

Diagonalizable

A set of eigenvectors that correspond to distinct eigenvalues is linear independent.

$$\det(A - tI_n) \quad \text{Factorization}$$

$$= (t - \lambda_1)^{\underline{m_1}} (t - \lambda_2)^{\underline{m_2}} \dots (t - \lambda_k)^{\underline{m_k}} (\dots \dots)$$



Diagonalizable

A set of eigenvectors that correspond to distinct eigenvalues is linear independent.

Eigenvalue: $\lambda_1 \quad \lambda_2 \quad \dots \quad \lambda_m$

Assume dependent

Eigenvector: $v_1 \quad v_2 \quad \dots \quad v_m$

➡ a contradiction

$$\mathbf{v}_k = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_{k-1} \mathbf{v}_{k-1}$$

$$A\mathbf{v}_k = c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 + \dots + c_{k-1} A\mathbf{v}_{k-1}$$

$$\lambda_k \mathbf{v}_k = c_1 \lambda_1 \mathbf{v}_1 + c_2 \lambda_2 \mathbf{v}_2 + \dots + c_{k-1} \lambda_{k-1} \mathbf{v}_{k-1}$$

$$- \lambda_k \mathbf{v}_k = c_1 \lambda_k \mathbf{v}_1 + c_2 \lambda_k \mathbf{v}_2 + \dots + c_{k-1} \lambda_k \mathbf{v}_{k-1}$$

(λ_k)

$$\mathbf{0} = c_1 (\lambda_1 - \lambda_k) \mathbf{v}_1 + c_2 (\lambda_2 - \lambda_k) \mathbf{v}_2 + \dots + c_{k-1} (\lambda_{k-1} - \lambda_k) \mathbf{v}_{k-1}$$

Not $c_1 = c_2 = \dots = c_{k-1} = 0$ ➡ Same eigenvalue ➡ a contradiction

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

$$\det(A - tI_n)$$

$$= (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k} (\dots)$$

Eigenvalue:

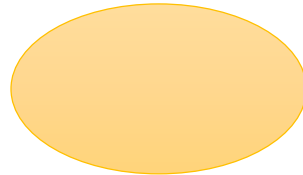
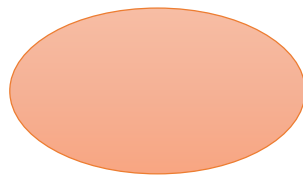
λ_1

λ_2

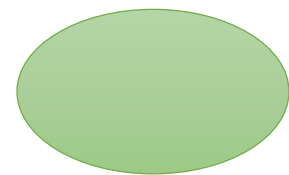
.....

λ_k

Eigenspace:



.....



Basis for λ_1

Basis for λ_2

Basis for λ_3




Independent Eigenvectors

You can't find more!

Diagonalizable - Example

- Diagonalize a given matrix $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

characteristic polynomial is $-(t + 1)^2(t - 3)$  eigenvalues: 3, -1

eigenvalue 3

$$B_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

eigenvalue -1

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$A = PDP^{-1}$,
where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

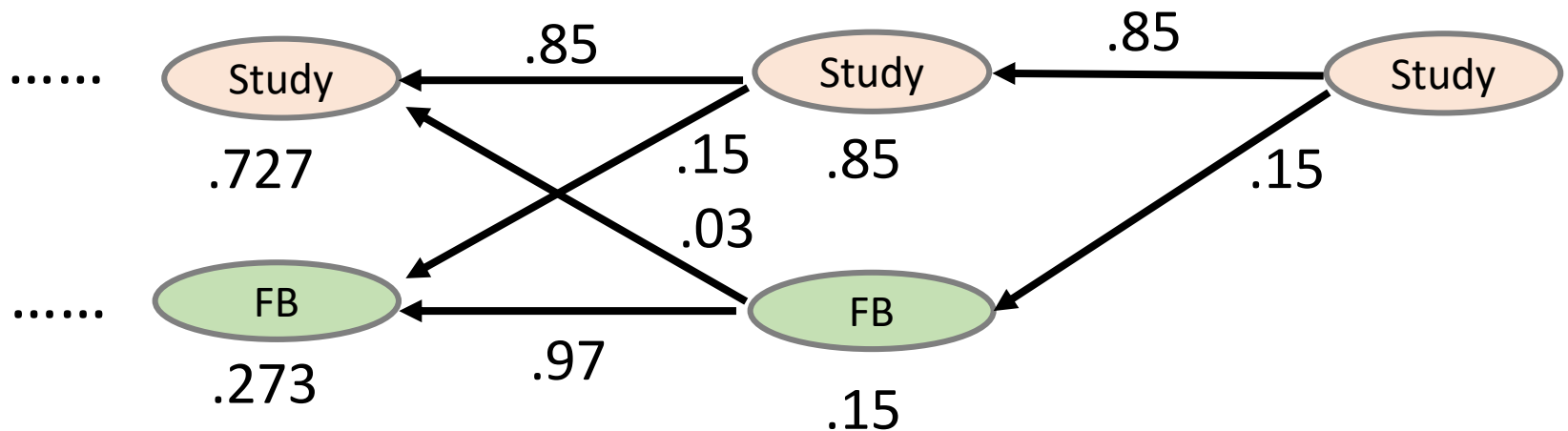
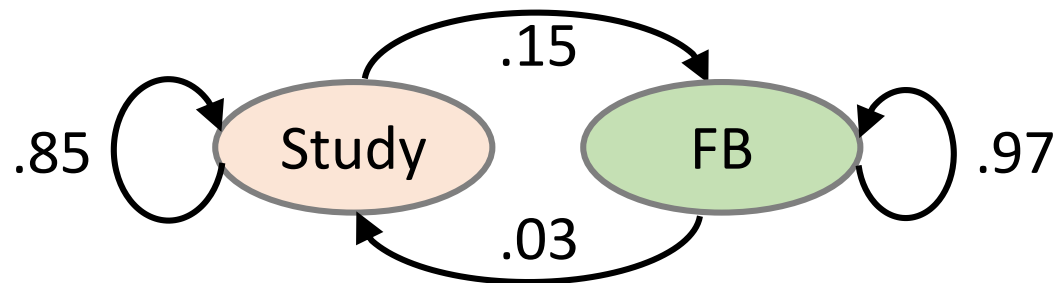
$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

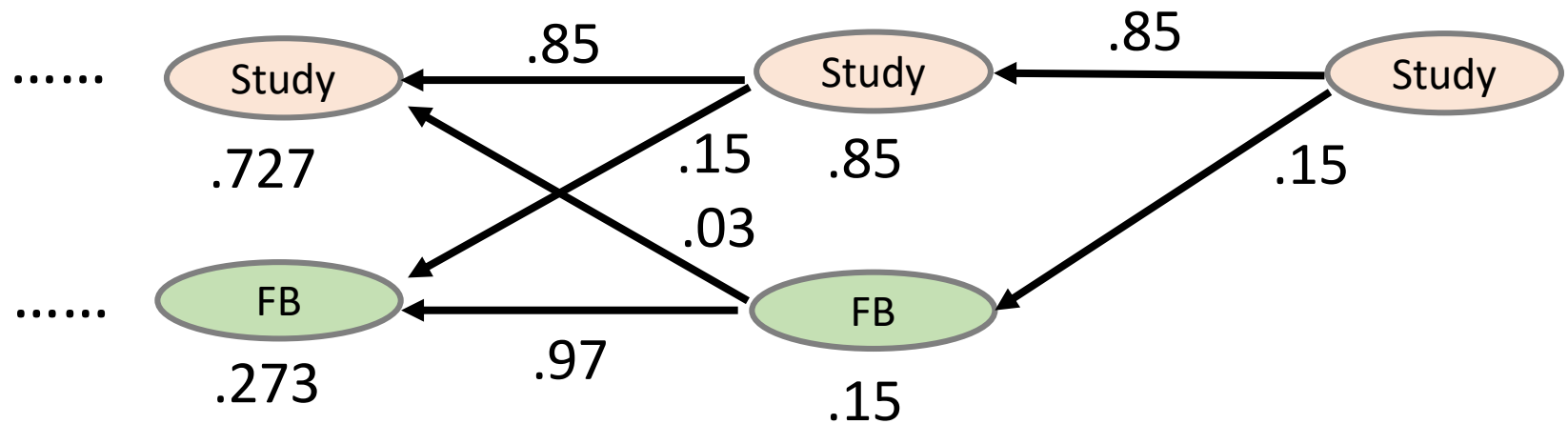
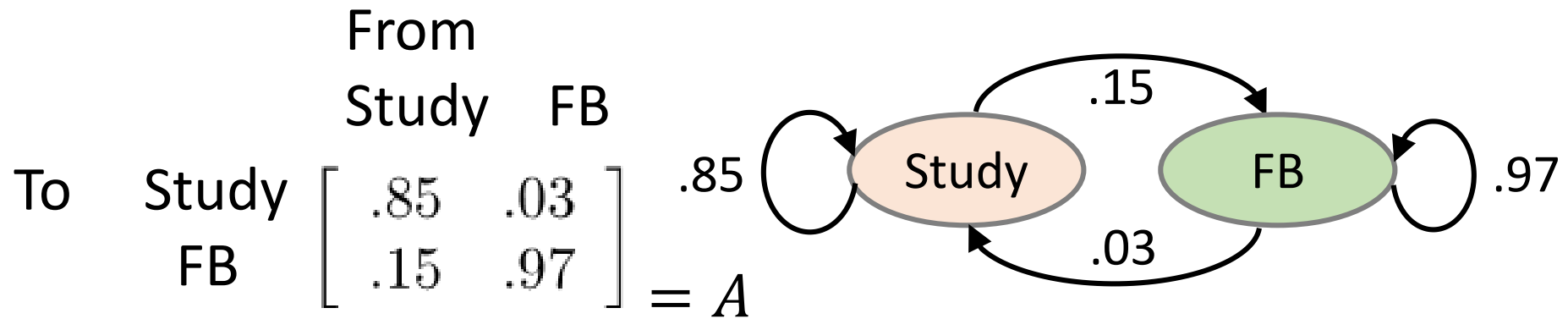
Application of Diagonalization

- If A is diagonalizable,

$$A = PDP^{-1} \longrightarrow A^m = PD^mP^{-1}$$

- Example:





$$\begin{bmatrix} .727 \\ .273 \end{bmatrix} \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} \begin{bmatrix} .85 \\ .15 \end{bmatrix} \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = PDP^{-1} \longrightarrow A^m = PD^mP^{-1}$$

Diagonalizable

- Diagonalize a given matrix $A = \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix}$

$$\det (A - tI_2)$$

$$\begin{array}{l} A - .82I_2 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ A - I_2 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -.2 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{p}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \end{array} \Rightarrow P = \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix} \text{ (invertible)}$$
$$D = \begin{bmatrix} .82 & 0 \\ 0 & 1 \end{bmatrix}$$

Application of Diagonalization

$$A = PDP^{-1} \text{ where } P = \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}, D = \begin{bmatrix} .82 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A^m &= PD^m P^{-1} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} (.82)^m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \\ &= \frac{1}{6} \begin{bmatrix} 1 + 5(.82)^m & 1 - (.82)^m \\ 5 - 5(.82)^m & 5 + (.82)^m \end{bmatrix} \end{aligned}$$

When $m \rightarrow \infty$,

$$A^m = \begin{bmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{bmatrix}$$

The beginning condition does not influence.

Test for a Diagonalizable Matrix

- An $n \times n$ matrix A is diagonalizable if and only if both the following conditions are met.

The characteristic polynomial of A factors into a product of linear factors.

$$\begin{aligned} \det(A - tI_n) & \text{Factorization} \\ & = (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \dots (t - \lambda_k)^{m_k} \text{(~~.....~~)} \end{aligned}$$

For each eigenvalue λ of A , the multiplicity of λ equals the dimension of the corresponding eigenspace.

Independent Eigenvectors

An $n \times n$ matrix A is diagonalizable

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The eigenvectors of A can form a basis for \mathbb{R}^n .

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$$\det(A - tI_n)$$

$$= (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \dots (t - \lambda_k)^{m_k} (\dots)$$

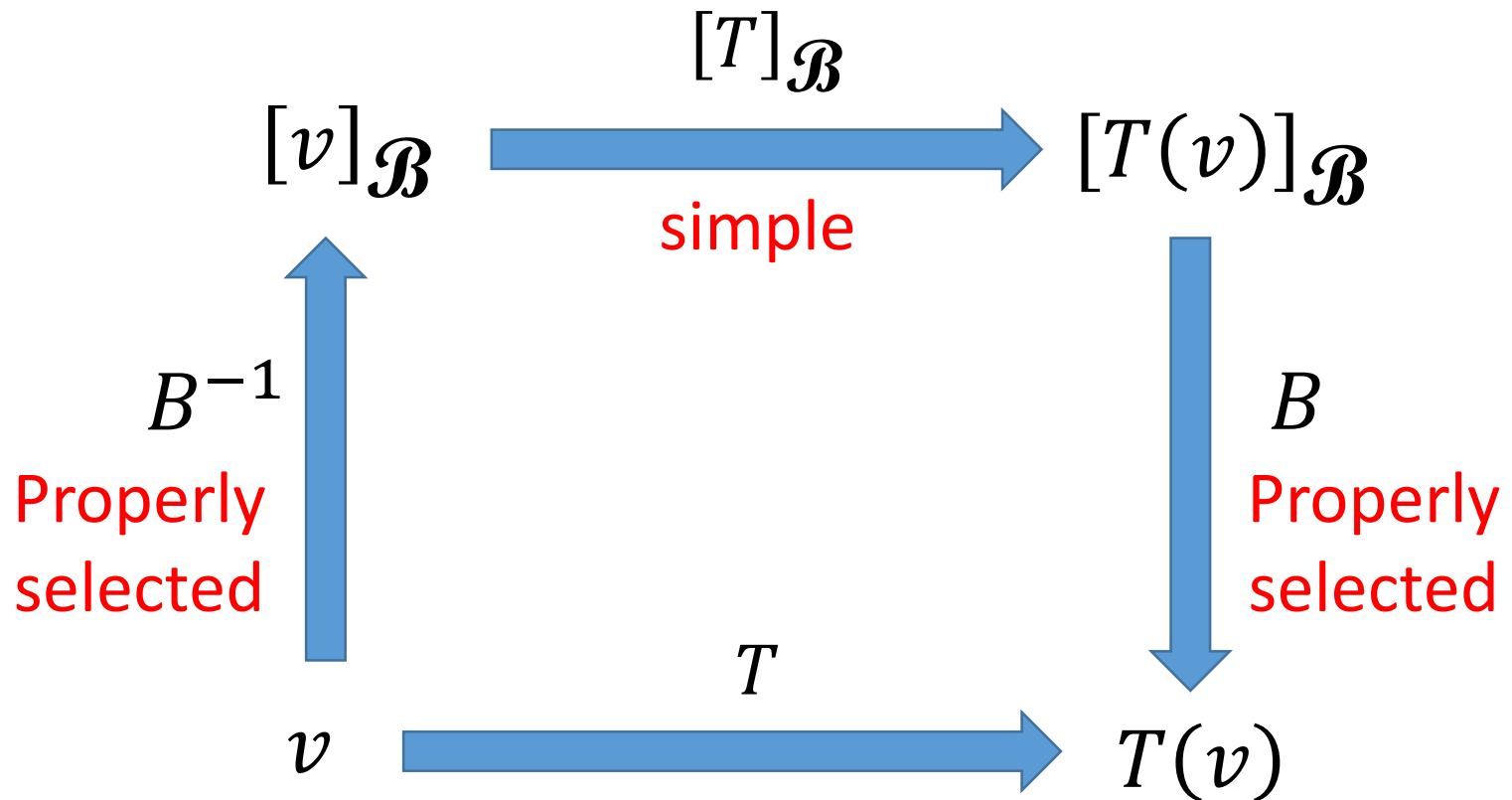
$$\text{Eigenvalue:} \quad \lambda_1 \quad \lambda_2 \quad \dots \quad \lambda_k$$

$$\text{Eigenspace:} \quad d_1 = m_1 \quad d_2 = m_2 \quad \dots \quad d_k = m_k$$

(dimension)

This lecture

- Reference: Chapter 5.4



Review

$$P = [p_1 \quad \cdots \quad p_n]$$

eigenvector

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

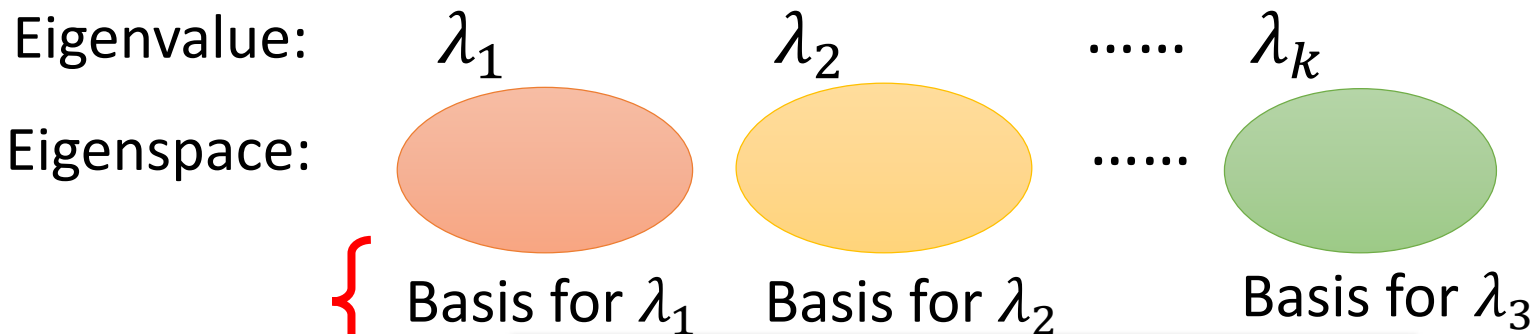
eigenvalue

An $n \times n$ matrix A is diagonalizable ($A = PDP^{-1}$)

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The eigenvectors of A can form a basis for \mathbb{R}^n .

$$\det(A - tI_n) = (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k} (\dots)$$



Independent Eigenvectors

Diagonalization of Linear Operator

• Example 1: $T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 8x_1 + 9x_2 \\ -6x_1 - 7x_2 \\ 3x_1 + 3x_2 - x_3 \end{bmatrix}$

The standard matrix is $A = \begin{bmatrix} 8 & 9 & 0 \\ -6 & -7 & 0 \\ 3 & 3 & -1 \end{bmatrix}$

\Rightarrow the characteristic polynomial is $-(t + 1)^2(t - 2)$

Eigenvalue -1:

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Eigenvalue 2:

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$\Rightarrow \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis of \mathcal{R}^3

Diagonalization of Linear Operator

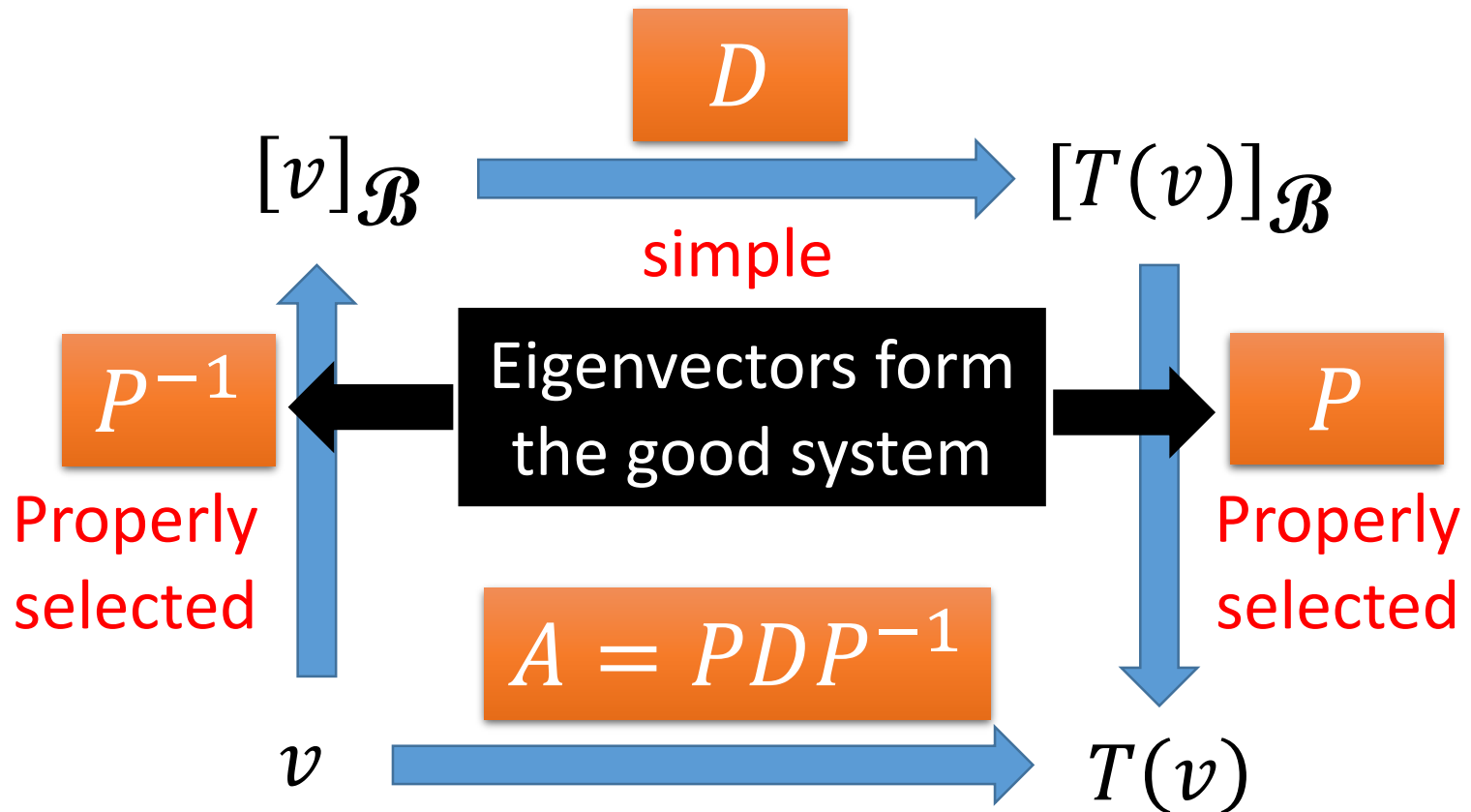
• Example 2: $T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} -x_1 + x_2 + 2x_3 \\ x_1 - x_2 \\ 0 \end{bmatrix}$

The standard matrix is $A = \begin{bmatrix} -1 & -t & 1 & 2 \\ 1 & -1 & -t & 0 \\ 0 & 0 & 0 & -t \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Diagonalization of Linear Operator

- If a linear operator T is diagonalizable



Diagonalization of Linear Operator

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 8x_1 + 9x_2 \\ -6x_1 - 7x_2 \\ 3x_1 + 3x_2 - x_3 \end{bmatrix} \quad \begin{matrix} -1: \\ \mathcal{B}_1 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \end{matrix} \quad \begin{matrix} 2: \\ \mathcal{B}_2 = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\} \end{matrix}$$

